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State Space Formulae for the Gap Computation

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Abstract

It is widely recognized that the computation of the gap metric is equivalent to a two block H^∞ problem or the norm computation of a two block "Hankel + Toeplitz" operator. However, it can also be characterized as the smallest singular value of a certain Toeplitz operator. We show here a simple computational method for finding such singular value and the gap between two plants using a state space skew Toeplitz type approach.

1 Introduction

The gap metric is a powerful tool for the study of system uncertainty [1]. In a remarkable paper [2], Georgiou shows that the gap is computable by solving a certain two block H^∞ problem. However, as shown in proposition 6 of [3], the gap can also be characterized as the smallest singular value of a certain Toeplitz operator. On the other hand, we have been investigating the so-called skew Toeplitz approach for one and two-block H^∞ problems [4, 5]. Here we apply the state space techniques employed in the computation of skew Toeplitz operator [4, 5, 7] to the gap computation problem.

2 Basic facts on gap computation

We consider the problem of the gap computation between two scalar real rational transfer functions $P_1(s)$ and $P_2(s)$. Suppose that they are given by the following minimal realizations:

$$P_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & d_i \end{array}\right], \quad i = 1, 2. \tag{1}$$

For the concrete characterization of the gap, we need the normalized coprime factorizations of the plants:

$$P_i(s) = N_i(s)D_i(s)^{-1}, N_i(s), D_i(s) \in \boldsymbol{RH}^\infty, \tag{2}$$

satisfying

$$N_i(s)^\sim N_i(s) + D_i(s)^\sim D_i(s) = 1. \tag{3}$$

With the normalized coprime factors (2), we can define the gap [3]. Let $\delta(P_1, P_2)$ denotes the gap between P_1 and P_2. The next result due to Georgiou [2] is the key to computing the gap:

Proposition 1 *The following equalities hold:*

$$\delta(P_1, P_2) = \max\left\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\right\}, \tag{4}$$

$$\vec{\delta}(P_1, P_2) = \inf_{Q \in H^\infty} \left\| \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} - \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Q \right\|_\infty. \tag{5}$$

This means that the directed gap $\vec{\delta}(P_1, P_2)$ can be computed as the solution to a two-block H^∞ problem:

$$\vec{\delta}(P_1, P_2) = \inf_{Q \in H^\infty} \left\| \begin{pmatrix} G - Q \\ J \end{pmatrix} \right\|_\infty, \tag{6}$$

where $G := D_2^\sim D_1 + N_2^\sim N_1$, $J := -N_2 D_1 + D_2 N_1$.

Let m be the Blaschke product which has the unstable poles of G as its zeros and let $\mathcal{H}(m) := H^2 \ominus mH^2$, and $W := mG$. Then $W \in \boldsymbol{RH}^\infty$. Using the commutant lifting theorem [6, 7], Georgiou [2] has shown that the directed gap (5) is equal to the norm of the two-block operator $Z : H^2 \to \begin{bmatrix} \mathcal{H}(m) \\ H^2 \end{bmatrix}$:

$$Z := \begin{bmatrix} P_{\mathcal{H}(m)} M_W|_{H^2} \\ P_{H^2} M_J|_{H^2} \end{bmatrix},$$

i.e.,

$$\vec{\delta}(P_1, P_2) = \|Z\|.$$

Then we can compute $\|Z\|$ by solving singular value equation of Z [5, 8]:

$$(Z^* Z - \gamma^2)\, x = 0. \tag{7}$$

In principle, we can apply the skew Toeplitz solution to the general two block problem. However, the special structure of the gap problem allows for further reduction as we see below.

Lemma 2 [3] *Let Θ_F denotes the Toeplitz operator with symbol $F \in \boldsymbol{RL}^\infty$ defined by*

$$\Theta_F := P_{H^2} M_F|_{H^2}. \tag{8}$$

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and $\rho_{\min}$ be the smallest singular value of Θ_G lying in the interval $[0,1]$. Then the directed gap $\vec{\delta}(P_1, P_2)$ is given by

$$\vec{\delta}(P_1, P_2) = \sqrt{1-\rho_{\min}^2}. \tag{9}$$

3 Computation of the singular values of Θ_G

Here, we will focus our attention on the computation of the singular values of Θ_G. We use a method similar to the one presented in [4, 5, 7] for the computation of singular values of skew Toeplitz operators. Let $\boldsymbol{C}_+$ and $j\boldsymbol{R}$ denotes the complex right half plane and the imaginary axis.

Assumption 1 *Let n denote the dimension of G. Assume that*

(a) *the equation*

$$G(s)^{\sim}G(s) - \rho^2 = 0 \tag{10}$$

has $2n$ mutually distinct roots $s_i, i = 1, \ldots, 2n$,

(b) *$s_i \in \boldsymbol{C}_+$ for $i = 1, \ldots, n_1$ and $s_i \in j\boldsymbol{R}$ for $i = n_1+1, \ldots, n_1+n_2$.*

With this assumption, we have the next theorem.

Theorem 3 *Under the Assumption 1 above, define a row vector ξ_i by*

$$\xi_i := \left[\rho B_1'(s_i I + \hat{A}_1')^{-1} \quad G^{\sim}(s_i)B_2'(s_i I + \hat{A}_2')^{-1}\right] \tag{11}$$

and a matrix Ξ_ρ

$$\Xi_\rho := \left[\xi_1' \; \xi_2' \cdots \xi_{n_1}'\right]'. \tag{12}$$

Then $\rho > 0$ is a singular value of Θ_G if and only if the following conditions hold:

(i) *$n_1 = n$, $n_2 = 0$, that is, none of the s_i lie on the imaginary axis.*

(ii) $\det \Xi_\rho = 0$.

Proof: Omitted for space limitation. □

4 Numerical example

Consider the directed gap between $P_1 = 1/(s+1)$ and $P_2 = 1/(s+2)$. We compute $\vec{\delta}(P_1, P_2)$ by two independent methods. One is the standard two-block approach and the other is the approach summarized above. Fig.1 (a) shows the Hankel operator norm with symbol GF_γ^{-1} versus γ where F_γ is a spectral factor of $\gamma^2 - J^{\sim}J$. The value of γ at which the Hankel norm $\|\Gamma_{GF_\gamma^{-1}}\|$ is equal to 1 gives $\vec{\delta}(P_1, P_2)$. Fig.1 (b) is the plot of $\det \Xi_\rho$ versus $\sqrt{1-\rho^2}$. The x coordinate of the point where the curve intersects $y = 0$ also indicates $\vec{\delta}(P_1, P_2)$.

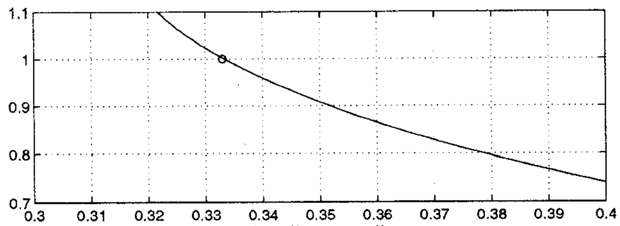

Figure 1: (a) $\|\Gamma_{GF_\gamma^{-1}}\|$ versus γ

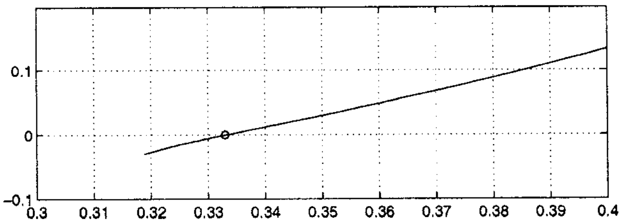

Figure 1: (b) $\det \Xi_\rho$ versus $(1-\rho^2)^{1/2}$

5 Conclusion

We have given a state space procedure for the computation of the singular values of a Toeplitz operator. Using this, we can compute the gap between two scalar plants via the one-block approach. Compared to the standard two-block approach, the computation is quite simple and requires neither spectral factorization nor Hankel norm computation.

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